Reasoning about possibilities: human reasoning violates all normal modal logics

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Abstract

Reasoning about possibilities is fundamental in daily life and in artificial intelligence. It is formalized in modal logics, of which there are infinitely many. Two experiments showed that individuals make inferences that are parsimonious about possibilities, and that they reject conclusions referring to possibilities that the premises do not support. Both sorts of inference contravene modal logics, i.e., the simplest system of modal logic and the infinite number of systems based on it.

Keywords: Modal logics, mental models, possibilities, reasoning, sentential connectives

Introduction

Human reasoning about possibilities is a major cognitive ability, and may be a precursor to reasoning about probabilities. All Indo-European and many other languages accordingly contain modal terms, such as *possibly* and *necessarily* (Palmer, 2001). These concepts are formalized in modal logics (Chellas, 1980; Fitting & Mendelsohn, 2012; Hughes & Cresswell, 1996; van Benthem, 2010), which are useful in software engineering (Kontchakov, 2010), artificial intelligence (Russell & Norvig, 2010), and philosophy, e.g., the logician Gödel used modal logic for an ontological proof of the existence of God (Benzmüller & Paleo, 2014). A crucial goal for cognitive scientists is to determine how naive individuals, who know nothing of logic, make modal inferences (Baron, 2008; Nickerson, 2010). Psychologists have studied how children develop notions of possibility (Piéraut-Le Bonniec, 1980; Sophian & Somerville, 1988; Shtulman 2007; 2009), and how adults deduce conclusions about possibilities from factual claims (Bell & Johnson-Laird, 1988; Bucciarelli & Johnson-Laird, 2005; Evans et al., 1999; Goldvarg & Johnson-Laird, 2000; Hinterecker, Knauff, & Johnson-Laird, 2016). But, they have seldom studied inferences from modal premises. One exception is a pioneering study of the relation between a simple modal logic and adolescent performance (Osherson, 1976). What complicates such studies is the existence of different concepts of possibility, such as those depending on logic alone (alethic), on knowledge (epistemic), on norms for action (deontic), on time (temporal). Another complication is the number of modal logics. There is a denumerable infinity of them (Fitting, 1972; see next section). So, the task of pinning down which modal logic, if any, underlies reasoning in daily life seems almost insuperable, which may explain the dearth of studies. Our concern is how individuals make deductions from modal premises to modal conclusions, and we compared a theory based on mental models with a logic that underlies infinitely many other modal logics. It is known as System K in honor of the logician Kripke, who showed how to formulate the semantics of modal logics in a way that relates directly to their different axioms (see below).

Modal logic in a nutshell

System K combines the sentential calculus, i.e., negation (*not*) and connectives, such as: conjunction (&), disjunction (or), implication (→) with two modal operators: possibly (◇) and necessarily (□). These two operators are interdefinable: if a proposition is possible, then it is not necessarily impossible. System K is based on a single modal axiom:

$$\Box (A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$$

That is, the necessity of proposition *A* implying proposition *B* in turn implies that the necessity of *A* implies the necessity of *B*. System K yields inferences that hold in all normal modal logics, and the effect of adding further axioms is to make new modal logics that yield additional inferences.

The proof that there are a denumerably infinite number of modal logics rests on the iteration of modal operators, as in: □□p, which asserts the necessity of necessarily p. But, perhaps whatever is necessary is necessarily necessary. Modality in daily life seldom concerns such issues: one can pile on modal operators merely for emphasis, e.g., *Perhaps* it is *possible* that it *may* rain. In a modal logic, however, it may be legitimate to collapse three modal operators into two, but not legitimate to collapse two modal operators into one. This principle can be arbitrarily extended upwards, so that it is legitimate to reduce n operators to n - 1, but no further, where n is a natural number. The result is a denumerable but infinite set of different modal logics.

The standard semantics for expressions in modal logic is in terms of 'possible worlds'. The basic idea is that if a proposition is possible then there is at least one world in the relevant set of possible worlds in which it is true. For example, if it is possible that it is raining in Yangon, then there is at least one possible world in which it is raining there. Logicians accordingly posit a set of possible worlds, W, which normally includes the actual world, so that for ◇p to be true, p is true in at least one world, w, which is a member of the relevant set of possible worlds W. Likewise, for □p to be true, p is true in every world w in the relevant set of possible worlds W. The relevant set of worlds in W is determined by a relation of *accessibility*. System K is the most general modal logic because it has no restrictions on the accessibility relation. System T is based on System K but it assumes that any world is accessible from itself, i.e., accessibility is a reflexive relation. This semantic assumption corresponds to the axiom (in the formal system for T):

$$\Box p \rightarrow p$$ [If p is necessary then p holds]

This assertion, which seems plausible, cannot be proved in K, because it lacks the axiom. As Kripke (1963) showed, different constraints on the accessibility relation yield the semantics for different axioms for modal logics. Human reasoners could rely on a tacit modal logic (Osherson, 1976), but an alternative theory is that they make mental simulations of possibilities (Johnson-Laird, 1983; Khemlani et al., 2013) in models isomorphic to the world (Johnson-Laird, Khemlani, & Goodwin, 2015; Ragni & Knauff, 2013; Ragni, Khemlani, & Johnson-Laird, 2014). According to this 'model' theory, three principles should apply to modal reasoning. First, all inferences are made in default of information to the contrary, and so reasoners abandon conclusions that facts contradict. This procedure is outside orthodox logic, which allows proofs of any conclusion whatsoever from such a contradiction. Hence, many so-called 'nonmonotonic' systems exist in cognitive science to allow them to make tentative inferences that they may subsequently withdraw (Marek & Truszynski, 2013; Orenes & Johnson-Laird, 2012; Ragni, Sonntag, & Johnson-Laird, 2016). The model theory merely gives preference to a subset of the premises in the case of a contradiction. Second, reasoners seek to minimize the number of mental models of distinct possibilities in order to reduce the load on memory. So, given the opportunity, they should conjoin separate possibilities into a single possibility. From the following sort of premise, for instance:

◇A & ◇B [*Possibly* A and *possibly* B]

they should construct the mental models that we represent in the following diagram in which each row denotes a possibility:

A B
. . .

The first row denotes a model of the possibility of *A* and *B* occurring together, but there are alternative possibilities, and the ellipsis is a model that allows for them without specifying their content. It follows that individuals should draw this conclusion from the premise (∴ denotes "therefore"):

∴ ◇ (A & B) [*Possibly:* A and B]

Here is an example of such an inference, which we used in our studies:

It is possible that Steven is in Madrid and it is possible that Emma is in Berlin.
Therefore, *it is possible that Steven is in Madrid and that Emma is in Berlin.*

Although the model theory predicts that individuals should draw this inference, it is invalid in all modal logics, from System K onwards, because Steven being in Madrid could imply that Emma is not in Berlin. In this case, both propositions in the premise remain possibilities and so the premise is true, but the conclusion is false. A corollary according to the model theory is that individuals should not draw the inference if the contents of the premise establish that one proposition implies the negation of the other. Third, reasoners take assertions such as disjunctions to refer to conjunctions of possibilities that hold in default of information to the contrary (Johnson-Laird et al., 2015). Hence, the disjunction:

A or B

refers by default to a conjunction of three exhaustive possibilities:

A		(Possibility 1: Only A holds)
	B	(Possibility 2: Only B holds)
A	B	(Possibility 3: A and B both hold)

When nothing in the premises supports one of the possibilities in such a conjunction, individuals should balk at the inference. For instance, they should not make the inference from the possibility of an exclusive disjunction to the possibility of an inclusive disjunction:

◇ (A xor B) [Possibly either A or B, but not both]
∴ ◇ (A or B) [Possibly A or B, or both]

where 'xor' denotes an exclusive disjunction in which both propositions, *A* and *B*, cannot hold together. Nothing in the premise supports their joint possibility, which is member of the conjunction of possibilities to which the conclusion refers. The inference would violate a fundamental principle of the model theory: a valid inference calls for the premises to support each possibility to which the conjunctive conclusion refers. Yet, the preceding inference is valid in modal logic, because the truth of the premise in System K guarantees the truth of the conclusion. The aim of our research was accordingly to make a crucial comparison between the model theory and modal logics. We carried out two experiments to investigate how participants reason with the modals *possible*, *necessary*, and *impossible*. We designed the experiments to contrast predictions from the model theory with those from normal modal logics.

Experiments

Experiment 1

This experiment was an exploratory study to examine a variety of modal operators: *possible*, *necessary*, and *impossible*. The participants evaluated 27 different representative inferences from a single modal premise to a single modal conclusion. The inferences used three different modal operators and three different connectives.

The pioneering study (Osherson, 1976) examined only inferences that are valid in modal logics K and T. In contrast, the present experiment also examined inferences that are invalid in modal logics. There were eight inferences that the model theory predicts that reasoners should accept even though they are invalid in System K, and two inferences that the theory predicts that reasoners should reject even though they are valid in System K.

Participants. We tested 53 logically naive participants (20 men, 23 women; M = 40.3 years).

Design and materials. The participants carried out 27 inferences consisting of 13 inferences valid in modal logic K and 14 inferences invalid in modal logic K (see Table 2). The inferences were presented in a different random order to each participant. Each inference consisted of a premise and a conclusion using one of the three modals *possible*, *necessary*, *impossible* and one of the connectives: *and*, *or _ or both* (inclusive-*or*), and *or _ but not both* (exclusive-*or*). A large scope modal is one that governs two clauses in an assertion, such as: *It is possible that Adam is in Berlin and Emma is in Boston*. Small scope modals govern each of the two clauses: *It is possible that Adam is in Berlin and it is possible that Emma is in Boston*. Two-thirds of the inferences had the same the modal in the premise and the conclusion, but with different scopes. One third of the inferences had premises and conclusions with different connectives such as exclusive 'or' inclusive 'or'. The inferences were about the location of individuals (common two-syllable proper names, such as *Adam*, and *Susan*) in well-known cities.

Procedure. The experiment was presented on an online website (Amazon's Mechanical Turk, hereafter MTurk). We took the usual precautions for such a procedure, e.g., the program checked that participants were native speakers of English, and it allowed only one participant from a given computer. The instructions explained that the task was not a test of intelligence or personality, but concerned general patterns of reasoning. The participants would read first an assertion ("a premise") then a second assertion ("a conclusion"), and for each pair they had to answer the question, "Does the premise imply that the conclusion is true?" The premise and conclusion were presented simultaneously. The participants responded by pressing one of two keys on their keyboards that were assigned to "Yes" and to "No." They could take as much time as they needed, but they had to try to answer correctly. This experiment and the subsequent one were implemented in Javascript. Before we carried out either experiment, we made extensive tests of this mTurk system to ensure that the responses were recorded reliably.

Results and discussion. Table 1 summarizes the results of the experiment in terms of the model theory's predictions and of modal logics' evaluations. Table 2 presents the results for each of the 27 inferences. They show that the participants' evaluations tended to corroborate the model theory. The r^2 correlation between the model theory's predictions and the results is 0.99, whereas the r^2 correlation between modal logic's evaluations and the results is 0.01. Overall, the participants made 80% of the model theory's predicted evaluations, but only 56% of modal logic's evaluations, which is hardly better than chance. This difference in the corroboration of the two accounts was highly robust (Wilcoxon test, $z = 6.1$, $p < .000001$). In particular, for those inferences in which the model theory and logic diverged, when the model theory predicted a 'Yes' evaluation, it occurred on 84% of trials, which was reliably greater than modal logics' predicted 'No' evaluation (16%; Wilcoxon, $z = 5.9$, $p < .000001$). Likewise, when the model theory predicted a 'No' evaluation, it occurred on 81% of trials, which was reliably greater than modal logics' predicted 'Yes' evaluation (19%; Wilcoxon, $z = 5.2$, $p < .000001$). The predictions of System K were only correct in so far as they coincided with those of the model theory.

Table 1. The percentages of 'Yes' evaluations of the inferences as valid, and of 'No' evaluations of the inferences as invalid in Experiment 1, as a function of the predictions of the model theory and of modal logics based on system K. The 53 participants acted as their own controls and evaluated 27 inferences based on contents concerning the locations of individuals, e.g., 'Adam is in Boston'. The participants' task was to answer the question: 'Does the premise imply that the conclusion is true?' and they made their evaluations by responding 'Yes' or 'No'. The percentages in bold are those for the mental model theory's predictions.

The predictions of the two accounts		The percentages of the participants' evaluations	
The model theory	Modal logics	Yes, the premise implies the conclusion	No, the premise does not imply the conclusion
Yes	No	**83**	17
Yes	Yes	**80**	20
No	Yes	19	**81**
No	No	27	**73**

Table 2. The percentages of 'Yes' evaluations of the inferences as valid, and of 'No' evaluations of the inferences as invalid for each of the 27 inferences in Experiment 1, as a function of the predictions of the model theory and of modal logics based on system K. The notation *Y*|*N* signifies that the model theory predicts a 'yes'

evaluation (the inference is valid), and its status in modal logic is ‘no’ (the inference is invalid); *or* denotes an inclusive disjunction, and *xor* denotes an exclusive disjunction. The inferences are ordered in blocks of three – each block containing one of the modals *possible*, *necessary*, and *impossible*.

Inference	Classification of inference	Mental models of possibilities	Percentages of inferences that the model theory predicts
◇A and ◇B ∴ ◇(A and B)	Y|N	A B . . .	92
□A and □B ∴ □(A and B)	Y|Y	A B	88
◇(A and B) ∴ ◇Aand ◇B	Y|Y	A B . . .	94
□(A and B) ∴ □A and □B	Y|Y	A B	88
◇A xor ◇B ∴ ◇(A xor B)	Y|Y	A B	90
◇A xor ◇B ∴ ◇(A and B)	N|N	A B	67
□A xor □B ∴ □ (A xor B)	Y|N	A B	92
◇(A xor B) ∴ ◇A xor ◇B	Y|N	A B	81
◇(A xor B) ∴ ◇(A or B)	N|Y	A B	79
□ (A xor B) ∴ □ (A and B)	N|N	A B	87
□ (A xor B) ∴ □ A xor □B	Y|N	A B	67
□ (A xor B) ∴ □ (A or B)	N|Y	A B	83
◇A or ◇B ∴ ◇(A or B)	Y|Y	A B A B	87
□A or □B ∴ □(A or B)	Y|Y	A B A B	92
◇(A or B) ∴ ◇(A xor B)	N|N	A B A B	65
◇(A or B) ∴ ◇A or ◇B	Y|Y	A B A B	90
□ (A or B) ∴ □ A or □B	Y|N	A B	85
□ (A or B) ∴ □ A xor □B	N|N	A B	83
¬◇A and ¬◇B ∴ ¬◇(A and B)	Y|Y	¬A ¬B . . .	83
¬◇A and ¬◇B ∴ ¬◇(A xor B)	Y|Y	¬A ¬B . . .	87
¬◇(A and B) ∴ ¬◇A and ¬◇B	Y|N	¬A ¬B . . .	87
¬◇(A xor B) ∴ ¬◇(A and B)	N|N	¬ A ¬B	62
¬◇(A xor B) ∴ ¬◇A xor B	Y|N	¬ A ¬B	83
¬◇(A xor B) ∴ ¬◇(A or B)	N|N	¬ A ¬B	73
¬◇A or ¬◇B ∴ ¬◇(A or B)	Y|N	¬ A ¬B	85
¬◇(A or B) ∴ ¬◇A or ¬◇B	Y|Y	¬ A ¬B	87
¬◇(A or B) ∴ ¬◇(A xor B)	Y|Y	¬ A ¬B	31

Experiment 2

This experiment systematically compared the model theory and modal logic using a single modal operator, *possible*. It examined four sorts of inference:

1. Y|N inferences, for which the model theory predicts ‘Yes, valid’, but they are invalid in modal logics.
2. Y|Y inferences, for which the model theory predicts ‘Yes, valid’, and they are valid in modal logics.
3. N|Y inferences, for which the model theory predicts ‘No, invalid’, but they are valid in modal logics.
4. N|N inferences, for which the model theory predicts ‘No, invalid’, and they are invalid in modal logics.

The premises and conclusions were based on three sentential connectives: *and*, *xor*, and *or*, and the modal operator: *possible*.

A typical Y|N inference was:

Premise: *It is possible that Tom is in Bristol or it is possible that Ann is in Cambridge, or both.*

Conclusion: *It is possible that Tom is in Bristol and it is possible that Ann is in Cambridge.*

Does the premise imply that the conclusion is true?

And a typical N|Y inference was:

Premise: *It is possible that Tom is in Bristol or it is possible that Ann is in Cambridge, but not both.*

Conclusion: *It is possible that Tom is in Bristol or it is possible that Ann is in Cambridge, or both.*

Does the premise imply that the conclusion is true?

Participants. We tested 51 logically naive participants (27m/24f, M = 38.9 years) .

Design, materials, and procedure. The participants acted as their own controls and evaluated 16 immediate inferences from a premise to a conclusion as in Study 1. The inferences consisted of 4 inferences of each the four sorts shown above: Y|N, Y|Y, N|Y, and N|N. They used only the modal operator *possible*, and systematically varied the connectives: *and*, *or*, and *xor*. And they were presented in a different random order to each participant. The contents of the inferences and the procedure were identical to those of the previous experiment.

Results and discussion. Table 3 summarizes the results of the experiment. They corroborated the model theory: The r^2 correlation between the model theory’s predictions and the results is 0.98, whereas the r^2 correlation between modal logic’s evaluations and the results is 0.05. Overall, the participants drew 83% of the model theory’s predicted conclusions but only 50% of modal logics’ evaluations (Wilcoxon, z = 6.2, p <

.0000001). When theory and logic diverged and the model theory predicted a 'Yes' evaluation, it occurred on 89% of trials, which was reliably greater than modal logics' predicted 'No' evaluation (11%; Wilcoxon, $z = 6.3$, $p < .0000001$). Likewise, when the model theory predicted a 'No' evaluation, it occurred on 76% of trials, which was reliably greater than modal logics' predicted 'Yes' evaluation (24%; Wilcoxon, $z = 5.5$, $p < .000001$).

Table 3. The percentages of 'Yes' evaluations and 'No' evaluations of the inferences in Experiment 2, as a function of the model theory's predictions and modal logics' predictions (based on System K). The 50 participants acted as their own controls and evaluated four inferences of each sort based on contents concerning the locations of individuals, e.g., 'Adam is in Boston'. The participants' task was to answer the question: 'Does the premise imply that the conclusion is true?' and they made their evaluations by responding 'Yes' or 'No'. The percentages in bold are those for the mental model theory's predictions.

The predictions of the two accounts		Percentages of participants' evaluations	
Model theory	Modal logics	'Yes'	'No'
Yes	No	**89**	11
Yes	Yes	**92**	8
No	Yes	24	**76**
No	No	25	**75**

General Discussion

Modal reasoning is ubiquitous in daily life, but rarely studied in cognitive science. In contrast, modal logics have flourished in philosophy, artificial intelligence, and in logic itself. As we outlined earlier, infinitely many modal logics derive from system K, which combines the sentential calculus with a single axiom dealing with modality. Human reasoning, as our experiments have shown, diverge from all these normal modal logics, and they do so in a way that the theory of mental models predicts. This model theory postulates that everyday inferences are always drawn in default of information to the contrary. When such information occurs, individuals are happy to explain the provenance of the inconsistency and to withdraw their conclusions (see, e.g., Johnson-Laird, Girotto, & Legrenzi, 2004; Khemlani & Johnson-Laird, 2012). The model theory also postulates that sentential connectives, such as *if*, *or,* and *xor*, refer to conjunctions of possibilities. Hence, an inference, such as:

The flaw is in the software or in the connection or both.
Therefore, *it is possible that the flaw is in the software.*

is readily accepted in daily life (Hinterecker, et al., 2016).

What the present studies have corroborated is the model theory's account of inferences that follow from premises containing modal operators. Such inferences depend on two main principles. First, inferences about a modality are parsimonious. That is, if a premise establishes some possibilities, they tend to be represented as co-occurring in the same model. Hence, individuals tend to make the following sort of inference (Experiment 1):

It is possible that Adam is in Berlin and it is possible that Anna is in Boston.
Therefore, it is possible that Adam is in Berlin and that Anna is in Boston.

When information or knowledge establishes that the two propositions cannot co-occur, as one of our unpublished studies shows, individual balk at the inference. Otherwise, humans make such parsimonious inferences even though they are invalid in modal logics.

Second, as a corollary of the principle that compounds refer to conjunctions of possibilities, everyday inferences are only deemed valid when the premises support all the possibilities to which their conjunctive conclusions refer. Hence, individuals tend to reject the following sort of inference (Experiment 2):

It is possible that Adam is in Bristol or it is possible that Anna is in Cambridge, but not both.
Therefore, *it is possible that Adam is in Bristol or it is possible that Anna is in Cambridge, or both.*

They do so according to the model theory, because a case to which the conclusion refers – it is possible that Adam is in Bristol and it is possible that Anna is in Cambridge – is not supported by either of the two possibilities to which the premise refers. Yet, the conclusion is valid in modal logic.

Proponents of modal logic might argue that our participants are merely wrong to make the previous sorts of inference, and that modal logic remains an arbiter of reasoning in daily life. We are sympathetic to this viewpoint, but regard it as mistaken. One reason is that the concept of possibility in daily life allows such assertions as:

Possibly it's raining and possibly it isn't.

It is a tautology in daily life, but not in system K. Moreover, in daily life, we all distinguish between factual possibilities, such as:

It may be raining

and counterfactual possibilities that occur when the facts are known, such as:

It isn't raining but it might have been.

The distinction is not drawn in modal logic, and so the preceding assertion is represented as:

(Not raining) & (◇ raining)

It, too, is false in modal logics. Yet, the counterfactual assertion above is not only sensible in daily life, it may well be true.

Our findings point to three conclusions: First, reasoning about possibilities in everyday life is a fundamental ability. Second, modal logics, despite their power and practical applications, diverge in fundamental ways from how naive individuals envisage possibilities. Third, the semantics of "possible worlds", which underlies modal logics, is too big to fit inside one's head (Partee, 1979), and so a potential alternative is to base it instead on the same finitary semantics as everyday probabilities (see Khemlani, Lotstein, & Johnson-Laird, 2015).

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